Bit Security of Quantum Key Search

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Quantum Threats and Cryptographic Key Sizes

Grover's algorithm **halves the bit security**

Example: Transition from AES-128 bits to AES-256 bits

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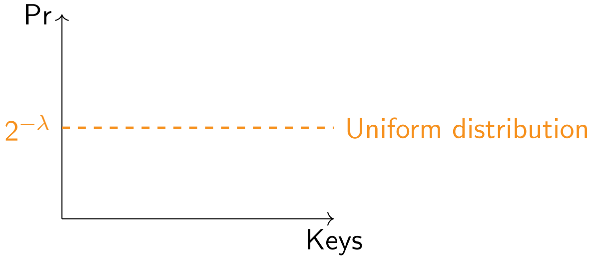
Example: Transition from AES-128 bits to AES-256 bits

Do we really get uniforms keys in practice?

From Statistical Distance to Bit Security



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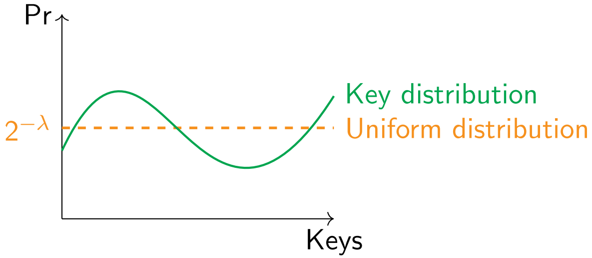

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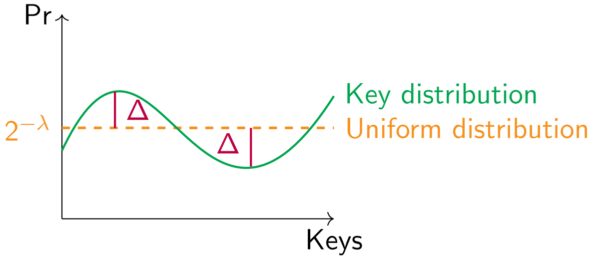

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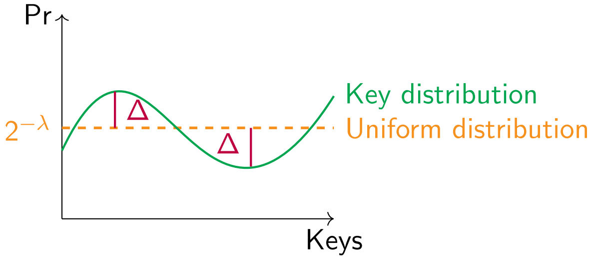
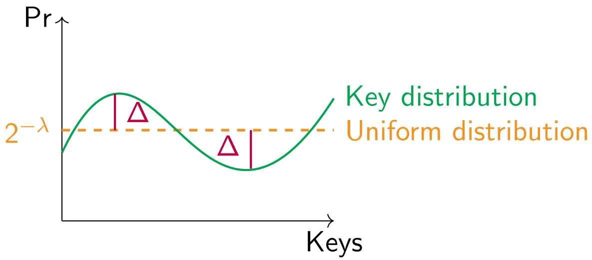


What is the reasonable range for the statistical distance?

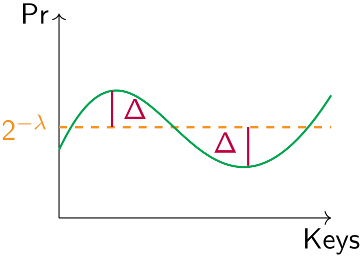
From Statistical Distance to Bit Security



What is the reasonable range for the statistical distance?

QKD $\Rightarrow$ Keys are close to uniform keys

Key Search in the Classical Setting



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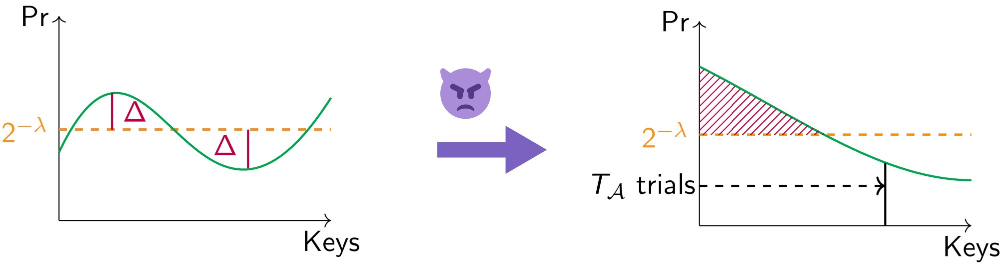

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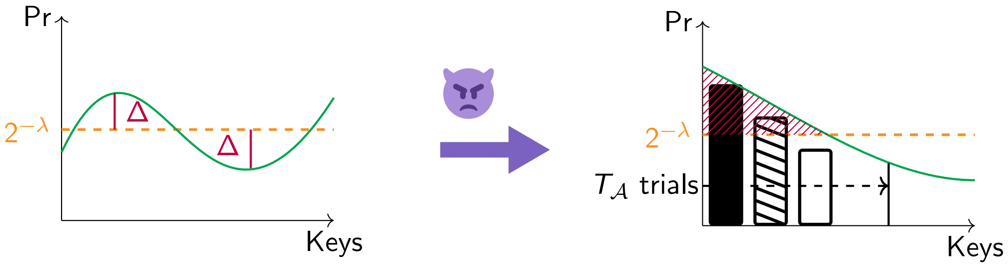

Strategy: Check the most probable keys first

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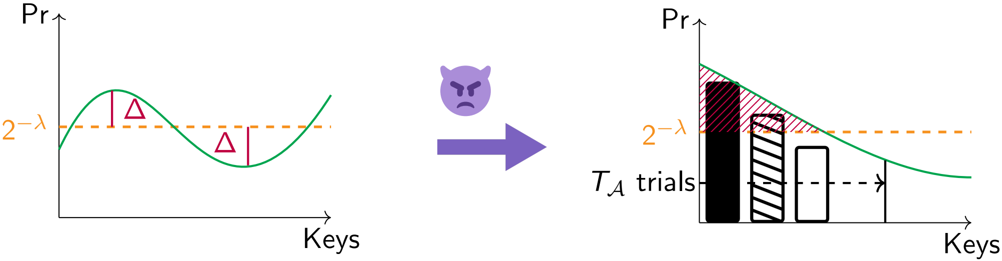

Strategy: Check the most probable keys first

Success probability of adversary

$$\epsilon_{\mathcal{A}} \leq T_{\mathcal{A}} \cdot 2^{-\lambda} + \Delta$$

Bit Security: Intuitively Definitions

A cryptographic system offers λ-bit security if any attacker is expected to require the effort of at least 2^λ to break the system.

Bit Security: Intuitively Definitions

$$Bs = \min_{\mathcal{A}} \log \frac{T_{\mathcal{A}}}{\epsilon_{\mathcal{A}}}$$

Bit Security: Key Definitions

Bit Security

- $Bs_{MW} = \min_{\mathcal{A}} \log \left(\frac{T_{\mathcal{A}}}{\mathsf{adv}_{MW}(\mathcal{A})} \right)$ [MW18]

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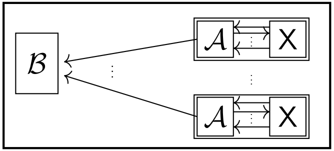

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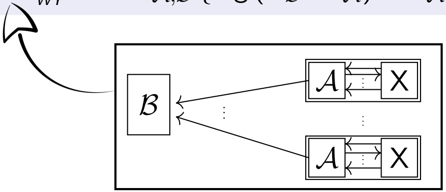
- $Bs_{WY} = \min_{\mathcal{A},\mathcal{B}} \{\log(N_{\mathcal{B}} \cdot T_{\mathcal{A}}) : \Pr_{\mathcal{A},\mathcal{B}} \geq 1 - \delta\}$ [WY21]



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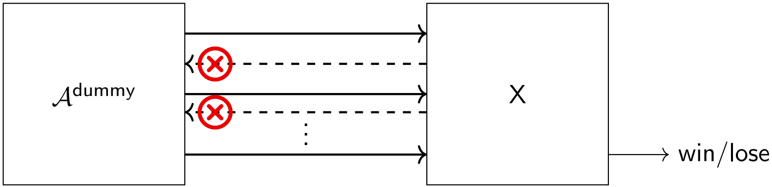
Bit Security via Observation Game

Baseline (Dummy) Adversary [Lee24]



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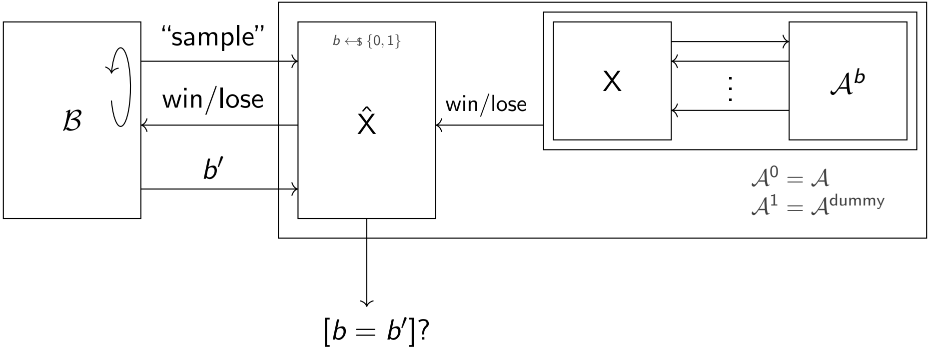

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👉 $$d_{\mathbf{Hell}}(\mathcal{P}, \mathcal{Q})^2 = \frac{1}{2} \sum_{x \in \Omega} \left(\sqrt{\mathcal{P}(x)} - \sqrt{\mathcal{Q}(x)} \right)^2$$

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Sample Complexity Bounds [Lee24]

$$\frac{1}{4\ln 2} \cdot \frac{\ln\left(\frac{1}{4\delta(1-\delta)}\right)}{d_{\mathrm{Hell}}(\mathcal{P},\mathcal{Q})^2} \leq N_\delta(\mathcal{P},\mathcal{Q}) \leq \frac{\ln\left(\frac{1}{2\delta}\right)}{d_{\mathrm{Hell}}(\mathcal{P},\mathcal{Q})^2}$$

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- $Bs_{Lee} = \min_{\mathcal{A}} \log\left(\frac{T_{\mathcal{A}}}{d_{\mathrm{Hell}}(\Pr^{G}_{\mathcal{A}}, \Pr^{G}_{\mathcal{A}_{\mathrm{dummy}}})^2}\right)$ [Lee24]

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Classic!

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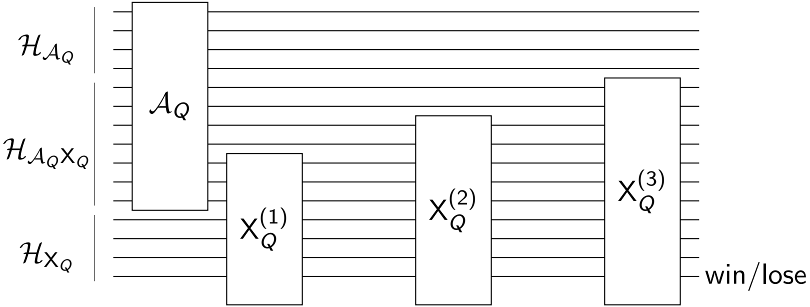
Proposed Hybrid Observation Game

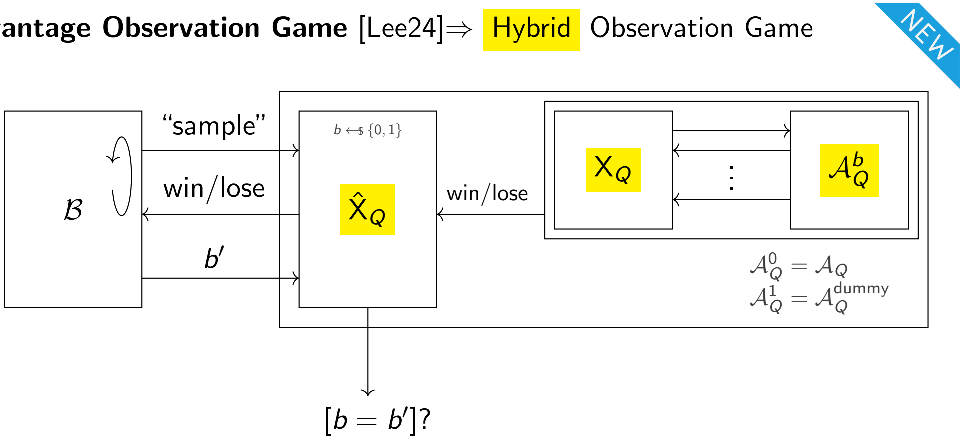
Baseline Adversary [Lee24] $\Rightarrow$ Quantum Dummy Adversary



NEW

Proposed Hybrid Observation Game

Advantage Observation Game [Lee24] ⇒ Hybrid Observation Game



Our Definition of Post-Quantum Bit Security

Definition (Post-Quantum Bit Security)

$$\mathsf{PQBS}^{\mathsf{G}_Q,\delta}_{\mathsf{Dem}}(\lambda) := \min_{\mathcal{A}_Q,\mathcal{B}} \left\{ \log(T_{\mathcal{A}_Q} \cdot N_{\mathcal{B}}) \ : \ \Pr^{\hat{\mathsf{G}}_Q}_{\mathcal{B}}(\lambda) \geq 1 - \delta(\lambda) \right\}$$

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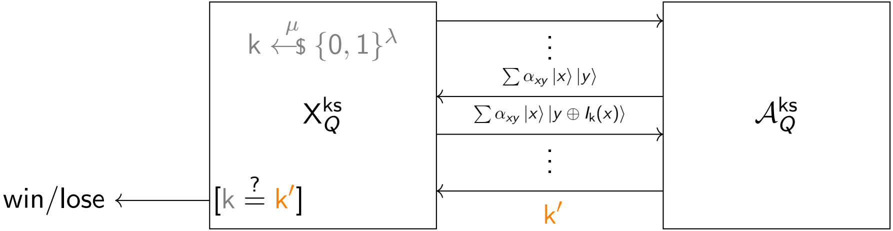
⇓

Definition (Hellinger Post-Quantum Bit Security)

$$\mathsf{PQBS}^{\mathsf{G}_Q}_{\mathsf{Hell}^2}(\lambda) = \min_{\mathcal{A}_Q} \log\left(\frac{T_{\mathcal{A}_Q}}{d_{\mathsf{Hell}}\left(\Pr^{\mathsf{G}_Q}_{\mathcal{A}_Q}(\lambda), \Pr^{\mathsf{G}_Q}_{\mathbf{D}[T_{\mathcal{A}_Q}]}(\lambda)\right)^2} \right)$$

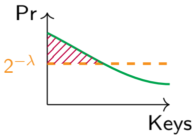
Quantum Key Search Game Model



Baseline and Success Probability

Quantum Dummy Adversary:

$$2^{-\lambda} \leq \Pr^{\mathsf{G}_Q^{\mathsf{ks},\mu,\Delta}}_{\mathbf{D}[T_Q]}(\lambda) \leq 2^{-\lambda} + \Delta$$



Independent of runtime !

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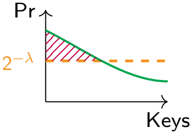

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Quantum Adversary:

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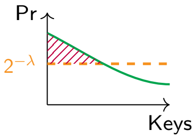

Independent of runtime!

Quantum Adversary:

$$T_{\mathcal{A}_Q^{\mathsf{ks}}}^2 \cdot 2^{-\lambda} \leq \Pr_{\mathcal{A}_Q^{\mathsf{ks}}}^{\mathsf{G}_Q^{\mathsf{ks},\mu,\Delta}}(\lambda)$$

[Mon11]: Montanaro. Quantum search with advice. 2011

[HSZ24]: He, Sun, Zhang. Quantum search with prior knowledge. 2024

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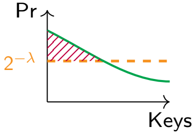

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Quantum Adversary:

$$T_{\mathcal{A}_Q^{\mathsf{ks}}}^2 \cdot 2^{-\lambda} \leq \Pr_{\mathcal{A}_Q^{\mathsf{ks}}}^{\mathsf{G}_Q^{\mathsf{ks},\mu,\Delta}}(\lambda) \leq 16\, T_{\mathcal{A}_Q^{\mathsf{ks}}}^2 \cdot 2^{-\lambda} + 2 \cdot \Delta$$

[BBHT98]: Boyer, Brassard, Høyer, Tapp. Tight bounds on quantum searching. 1998
[Zal99]: Zalka. Grover's quantum searching algorithm is optimal. 1999

Bounds with $\Delta \leq 2^{-\lambda}$

Assumption: Adversary runtime is $T_{\mathcal{A}_Q^{\mathsf{ks}}} \leq 2^{\lambda/2}$

Lower-Upper Bound

$$\min_{\mathcal{A}_Q^{\mathsf{ks}}}(\lambda - \log T_{\mathcal{A}_Q^{\mathsf{ks}}} - 5) \;\leq\; \mathsf{PQBS}^{\mathsf{G}_Q}_{\mathsf{Hell}^2}(\lambda) \;\leq\; \min_{\mathcal{A}_Q^{\mathsf{ks}}}(\lambda - \log T_{\mathcal{A}_Q^{\mathsf{ks}}} + 3)$$

Implications:

- Bounds match up to a constant number of bits
- For $T_{\mathcal{A}_Q^{\mathsf{ks}}} = 2^{\lambda/2} \Rightarrow \mathsf{PQBS}^{\mathsf{G}_Q}_{\mathsf{Hell}^2}(\lambda) \approx \lambda/2$
- No further gain when $\Delta < 2^{-\lambda}$

Bounds for $2^{-\lambda} \leq \Delta \leq 2^{-\lambda/2}$

Example: Let $\Delta = 2^{-\lambda/2}$

Lower-Upper Bound

$$\min_{\mathcal{A}_Q^{\mathsf{ks}}}(\lambda/2 + \log T_{\mathcal{A}_Q^{\mathsf{ks}}} - 5) \;\leq\; \mathsf{PQBS}^{\mathsf{G}_Q}_{\mathsf{Hell}^2}(\lambda) \;\leq\; \min_{\mathcal{A}_Q^{\mathsf{ks}}}(\lambda - \log T_{\mathcal{A}_Q^{\mathsf{ks}}} + 3)$$

Implications:

- $T_{\mathcal{A}_Q^{\mathsf{ks}}} = 1 \Rightarrow$ Lower bound offers at least $\lambda/2$ bit security
- $T_{\mathcal{A}_Q^{\mathsf{ks}}} = 2^{\lambda/2} \Rightarrow$ Bounds matching

Example: When $\Delta = 2^{-\lambda}$

$$\Rightarrow \mathsf{PQBS}^{\mathsf{G}_Q}_{\mathsf{Hell}^2}(\lambda) \approx \lambda - \log T_{\mathcal{A}_Q^{\mathsf{ks}}}$$

Bounds when $\Delta > 2^{-\lambda/2}$

Lower-Upper Bound

$$\min_{\mathcal{A}_Q^{\mathsf{ks}}}(\log T_{\mathcal{A}_Q^{\mathsf{ks}}} - \log \Delta - 5) \leq \mathsf{PQBS}_{\mathsf{Hell}^2}^{\mathsf{G}_Q}(\lambda) \leq \min_{\mathcal{A}_Q^{\mathsf{ks}}}(\lambda - \log T_{\mathcal{A}_Q^{\mathsf{ks}}} + 3)$$

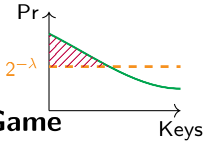
Example: For $\Delta = 2^{-\lambda/4}$ and $T_{\mathcal{A}_Q^{\mathsf{ks}}} = 1$

$$\min_{\mathcal{A}_Q^{\mathsf{ks}}}(\lambda/4 - 5) \leq \mathsf{PQBS}_{\mathsf{Hell}^2}^{\mathsf{G}_Q}(\lambda) \leq \min_{\mathcal{A}_Q^{\mathsf{ks}}}(\lambda + 3)$$

Interpretation:

- Notably decreased lower bound
- Upper bound is not tight compared with the lower bound
- Worst-case testing only one key

Takeaways



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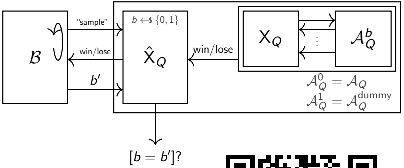
- ✓ Proposed **a definition** for PQBS based on **Hybrid Observation Game**
- ✓ Fixed **bounds** for the PQBS based on a **Quantum Key Search Game**
- ✓ Gave the interpretation of the bounds:

> $\Rightarrow \Delta < 2^{-\lambda}$, not any advantage for the bit security
> $\Rightarrow 2^{-\lambda} \leq \Delta \leq 2^{-\lambda/2}$, $\Delta = 2^{-\lambda}$ is conservative choice
> $\Rightarrow \Delta > 2^{-\lambda/2}$, is unclear as result for the bit security



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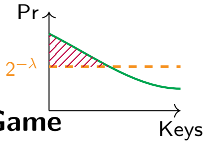
pre-proceeding version **SAC 2025**

Takeaways



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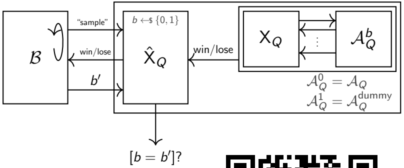

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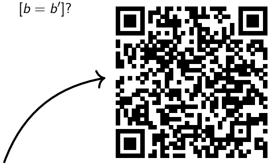
Thank you !



pre-proceeding version 🔗 **SAC 2025**

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Extra Slides

Upper Bound for the Post-Quantum Bit Security I

- $d_{\text{Hell}}(\mathcal{P}, \mathcal{Q})^2 = 1 - \sqrt{\epsilon_{\mathcal{P}} \cdot \epsilon_{\mathcal{Q}}} - \sqrt{(1-\epsilon_{\mathcal{P}}) \cdot (1-\epsilon_{\mathcal{Q}})}$

Use upper bounds on $\epsilon_{\mathcal{P}}$ and $\epsilon_{\mathcal{Q}}$ to lower-bound the distance:

$$\epsilon_{\mathcal{P}} \cdot \epsilon_{\mathcal{Q}} = \left(16\, T^2_{\mathcal{A}^{\text{ks}}_Q} \cdot 2^{-\lambda} + 2\Delta\right) \cdot \left(2^{-\lambda} + \Delta\right)$$

$$(1-\epsilon_{\mathcal{P}}) \cdot (1-\epsilon_{\mathcal{Q}}) = \left(1 - T^2_{\mathcal{A}^{\text{ks}}_Q} \cdot 2^{-\lambda}\right) \cdot \left(1 - 2^{-\lambda}\right)$$

- **Case 1:** $\Delta \leq 2^{-\lambda},\ T_{\mathcal{A}^{\text{ks}}_Q} \geq 48$:

$$d_{\text{Hell}}\left(\Pr^{\mathsf{G}_Q}_{\mathcal{A}_Q}(\lambda), \Pr^{\mathsf{G}_Q}_{\mathbf{D}[T_{\mathcal{A}^{\text{ks}}_Q}]}(\lambda)\right)^2 \geq \frac{1}{8} T^2_{\mathcal{A}^{\text{ks}}_Q} \cdot 2^{-\lambda} \quad \text{[Lower bound]}$$

$$\Rightarrow \text{[Upper bound]} \quad \mathsf{PQBS}^{\mathsf{G}_Q}_{\text{Hell}^2}(\lambda) \leq \min_{\mathcal{A}^{\text{ks}}_Q}(\lambda - \log T_{\mathcal{A}^{\text{ks}}_Q} + 3).$$

Upper Bound for the Post-Quantum Bit Security II

- **Case 2:** $2^{-\lambda} < \Delta \leq \frac{1}{48^2} T^2_{\mathcal{A}_Q^{\mathsf{ks}}} \cdot 2^{-\lambda}$, with $\sqrt{\gamma} \leq \frac{1}{48}$, then:

$$d_{\mathsf{Hell}}\left(\Pr^{\mathsf{G}_Q}_{\mathcal{A}_Q}(\lambda), \Pr^{\mathsf{G}_Q}_{\mathbf{D}[T_{\mathcal{A}_Q^{\mathsf{ks}}}]}(\lambda)\right)^2 \geq \frac{1}{8} T^2_{\mathcal{A}_Q^{\mathsf{ks}}} \cdot 2^{-\lambda} \quad \text{[Lower bound]}$$

$$\Rightarrow \text{[Upper bound]} \quad \mathsf{PQBS}^{\mathsf{G}_Q}_{\mathsf{Hell}^2}(\lambda) \leq \min_{\mathcal{A}_Q^{\mathsf{ks}}}(\lambda - \log T_{\mathcal{A}_Q^{\mathsf{ks}}} + 3).$$

Lower Bound for the Post-Quantum Bit Security I

- Hellinger Distance:

$$d_{\mathsf{Hell}}(\mathcal{P},\mathcal{Q})^2 \leq d_{\mathsf{TV}}\left(\Pr^{G_Q}_{\mathcal{A}_Q}(\lambda), \Pr^{G_Q}_{\mathbf{D}[T_{\mathcal{A}_Q^{\mathsf{ks}}}]}(\lambda)\right) \leq 16\,T^2_{\mathcal{A}_Q^{\mathsf{ks}}} \cdot 2^{-\lambda} + 2\cdot\Delta$$

- **Case 1:** $\Delta \leq T^2_{\mathcal{A}_Q^{\mathsf{ks}}} \cdot 2^{-\lambda}$

$$d_{\mathsf{Hell}}\left(\Pr^{G_Q}_{\mathcal{A}_Q}(\lambda), \Pr^{G_Q}_{\mathbf{D}[T_{\mathcal{A}_Q^{\mathsf{ks}}}]}(\lambda)\right)^2 \leq 18\,T^2_{\mathcal{A}_Q^{\mathsf{ks}}} \cdot 2^{-\lambda} \quad \text{[Upper bound]}$$

$$\Rightarrow \text{[Lower bound]} \quad \mathsf{PQBS}^{G_Q}_{\mathsf{Hell}^2}(\lambda) \geq \min_{\mathcal{A}_Q^{\mathsf{ks}}}(\lambda - \log T_{\mathcal{A}_Q^{\mathsf{ks}}} - 5)$$

Lower Bound for the Post-Quantum Bit Security II

- **Case 2:** $\Delta > T^2_{\mathcal{A}_Q^{\mathsf{ks}}} \cdot 2^{-\lambda}$

$$d_{\mathsf{Hell}}\left(\Pr^{\mathsf{G}_Q}_{\mathcal{A}_Q}(\lambda), \Pr^{\mathsf{G}_Q}_{\mathbf{D}[T_{\mathcal{A}_Q^{\mathsf{ks}}}]}(\lambda)\right)^2 < 18 \cdot \Delta \quad [\text{Upper bound}]$$

$$\Rightarrow [\text{Lower bound}] \quad \mathsf{PQBS}^{\mathsf{G}_Q}_{\mathsf{Hell}^2}(\lambda) \geq \min_{\mathcal{A}_Q^{\mathsf{ks}}}(\log T_{\mathcal{A}_Q^{\mathsf{ks}}} - \log \Delta - 5)$$

If $\Delta = \gamma \cdot T^2_{\mathcal{A}_Q^{\mathsf{ks}}} \cdot 2^{-\lambda}$, with $\gamma > 1$, then

$$\mathsf{PQBS}^{\mathsf{G}_Q}_{\mathsf{Hell}^2}(\lambda) \geq \min_{\mathcal{A}_Q^{\mathsf{ks}}}(\lambda - \log T_{\mathcal{A}_Q^{\mathsf{ks}}} - \log \gamma - 5)$$

QKD Error Parameters

Error Decomposition in QKD
[RK05, MQR09, TGR12, MCIT15, TL17, BGKE20, PR22, LYW+21, RW23]:

$$\varepsilon = \varepsilon_{\text{correct}} + \varepsilon_{\text{secure}}$$

$\Rightarrow \varepsilon_{\text{correct}}$: Not **Identical keys** for both parties.
$\Rightarrow \varepsilon_{\text{secure}}$: Adversary has **information about key**.

- In [RK05, MQR09, TL17, BGKE20, PR22, RW23] **trace distance** $\approx$ statistical distance.

Discussion Points:

- $\varepsilon_{\text{secure}}$ corresponds to our **statistical distance**.
- Choosing $\varepsilon_{\text{correct}} = \varepsilon_{\text{secure}}$ is **cryptographically problematic**:
  - Correctness is verifiable; secrecy is not.
  - So: $\varepsilon_{\text{secure}} \ll \varepsilon_{\text{correct}}$ is often preferable.
  - The more realistic option is maybe $\varepsilon = 10^{-5}$ by [ZLR+22].

Privacy Amplification and Bit Security

Impact of $\varepsilon_{\text{secure}}$ on Privacy Amplification:

$$\textbf{Cut bits} \approx 2\log\frac{1}{\varepsilon_{\text{secure}}}$$

Example: AES-256 Key

- For $\varepsilon_{\text{secure}} = 2^{-40}$: need **80** extra bits.
  $\Rightarrow 256 + 80 = 336$ reconciled bits for $\varepsilon_{\text{secure}} = 2^{-40}$.
- For $\varepsilon_{\text{secure}} = 2^{-256}$: need **512** extra bits.
  $\Rightarrow 256 + 512 = 768$ reconciled bits for $\varepsilon_{\text{secure}} = 2^{-256}$.

Thoughts:

- The value ε in literature is maybe **optimistic**.
- The security level depends sensitively on $\varepsilon_{\text{secure}}$, **not just the sum**.