Multi-precision PMNS with CIOS reduction

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Modular Arithmetic

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  - Discrete Logarithm Problem
  - $\sim 1024, 2048, \ldots$ bits

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- Isogeny-based Cryptography
  - Explicit Isogeny Finding Problem
  - $\sim 512, 1024, \ldots$ bits

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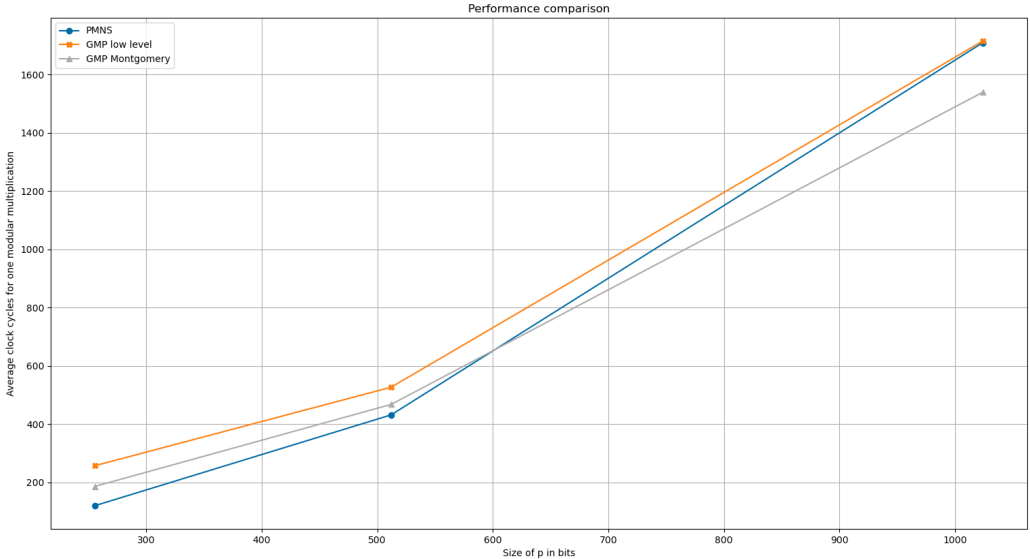
- General Methods
  - Montgomery reduction, Barret reduction, etc.
  - Representation systems (RNS, **PMNS**)

Definition

What is a PMNS?

- Polynomial Modular Number System (PMNS) is a non-positional number system based on polynomial operations.
- Proposed as an extension of Mersenne numbers to generic primes by J.-C. Bajard, L. Imbert and T. Plantard in 2004.
- Fast modular operations
- Very adapted to parallel processing (SIMD and Multi-Threading).

Performances on small sizes



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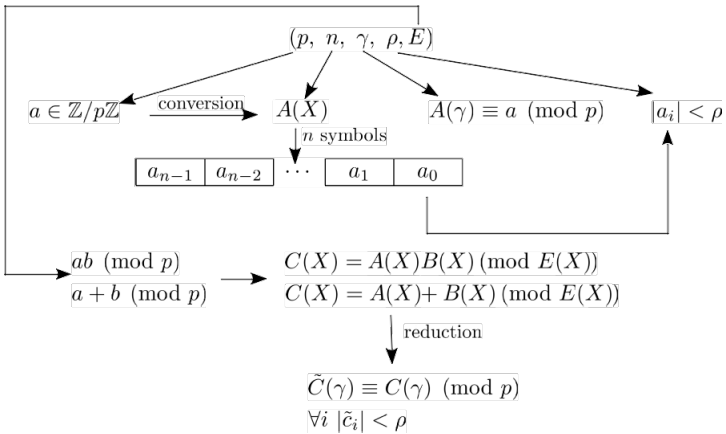
Polynomial Modular Number System



$E \in \mathbb{Z}[X]$ with $\deg(E) = n$ and $E(\gamma) \equiv 0 \pmod p$. Here we'll take $E(X) = X^n - \lambda$, $\lambda \in \mathbb{Z}^*$.

Toy example

$\mathcal{B} = (p = 11, n = 3, \gamma = 7, \rho = 2, E = X^3 - 2)$

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- $C''(\gamma) \equiv C''(7) \equiv -7 \equiv 4 \pmod{11}$
- $ab \equiv 8 \times 6 \equiv 48 \equiv 4 \pmod{11}$

Problems with the Internal Reduction

($p = 726589512580075827165577815940205655126075658084463158895150129844038996753 09$, $n = 5$, $\gamma = 472336245400502276405725161213198813429259421923382283164897368044854918329 57$, $\rho = 18014398509481984$, $E(X) = X^5 - 2$)

- $A(X) = -16418323151436339X^4 + 276252764540687X^3 + 7437842018938432X^2 + 15470279641088753X - 2872875599240331$
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Montgomery Modular Multiplication

Algorithm 1 Montgomery Modular Multiplication

Require: m the modulus, $a \in \mathbb{Z}/m\mathbb{Z}$, $b \in \mathbb{Z}/m\mathbb{Z}$, $\phi = 2^h$ with $h \in \mathbb{N}^*$ such that $2^{h-1} < m < 2^h$.

Ensure: $c \equiv ab\phi^{-1} \bmod m$, with $c \in \mathbb{Z}/m\mathbb{Z}$

1: $c \leftarrow a \times b$
2: $q \leftarrow c \times m^{-1} \bmod \phi$
3: $c \leftarrow (c - q \times m)/\phi$ # Exact division
4: **return** c

$(c \times m^{-1} \bmod \phi) \times m \equiv c \pmod{\phi}$ so $c - q \times m \equiv 0 \pmod{\phi}$.

Montgomery Lattice Reduction

Adaptation by C. Negre and T. Plantard in 2008.

Algorithm 2 Montgomery-like Polynomial Multiplication

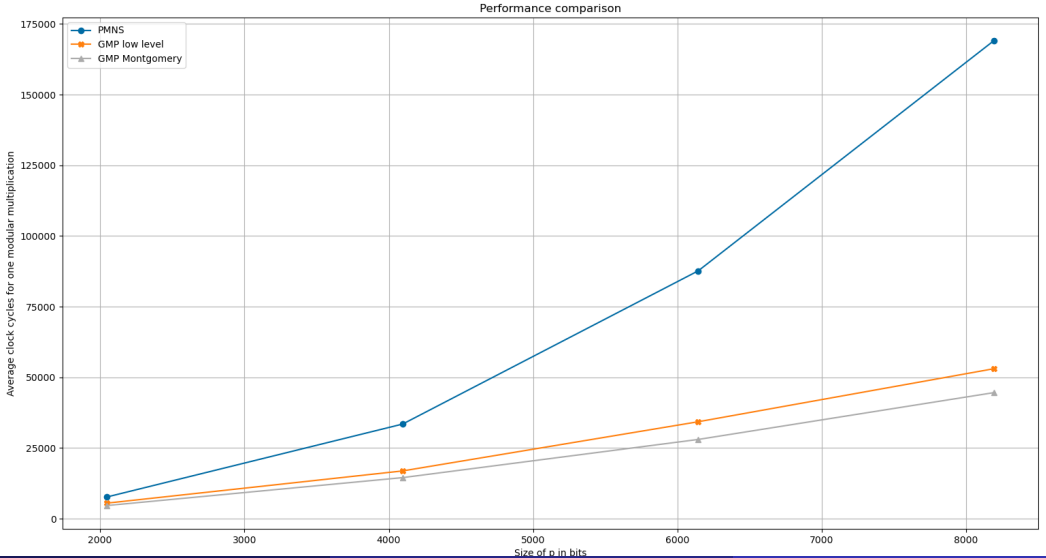
Require: $\mathcal{B} = (p,\ n,\ \gamma,\ \rho, E)$ a PMNS, $A, B \in \mathcal{B} \times \mathcal{B}$, $M \in \mathbb{Z}[X]$ such that $M(\gamma) \equiv 0 \pmod{p}$, $\phi = 2^h$ with $h \in \mathbb{N}^*$.

Ensure: $C(\gamma) \equiv A(\gamma)B(\gamma)\phi^{-1} \bmod p$, with $C' \in \mathcal{B}$

1: $C \leftarrow A(X) \times B(X) \pmod{E(X)}$
2: $Q \leftarrow C(X) \times M^{-1}(X) \pmod{E(X), \phi}$
3: $T \leftarrow Q(X) \times M(X) \pmod{E(X)}$
4: $C \leftarrow (C(X) - T(X))/\phi$ # Exact division
5: return C

For performance reasons ϕ is taken as $\phi = 2^{64}$.

Performance on large sizes



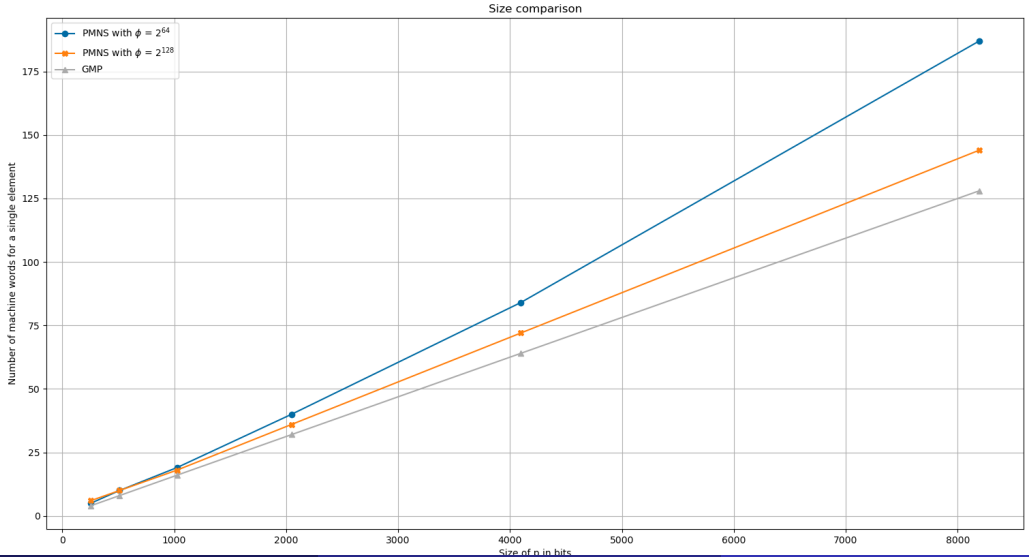
Internal Reduction Matrix

We can view multiplication by M mod $E(X) = X^n - \lambda$ as multiplication by

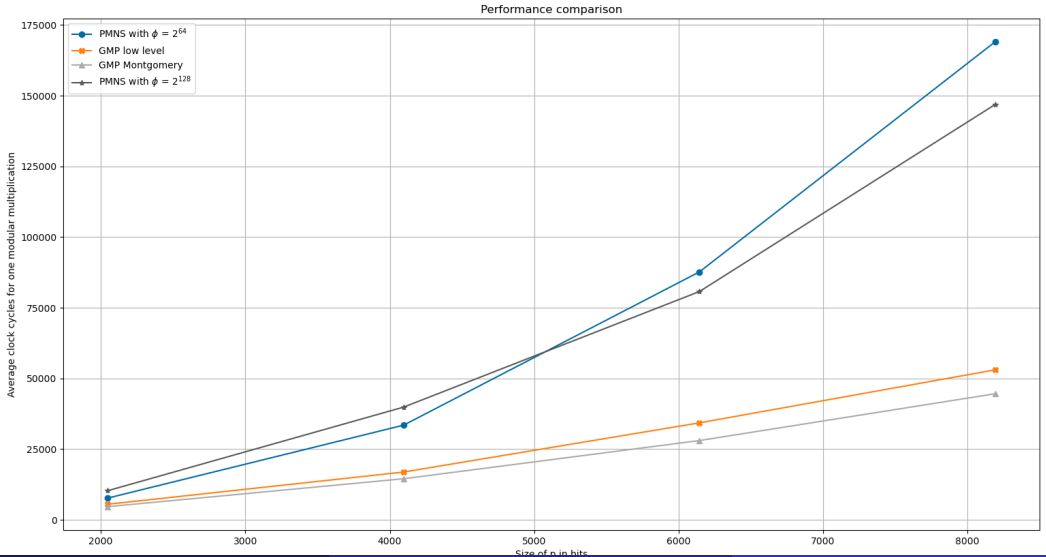
$$\mathcal{M} = \begin{pmatrix} m_0 & m_1 & \dots & m_{n-1} \\ \lambda m_{n-1} & m_0 & \dots & m_{n-2} \\ \vdots & \ddots & \ddots & \vdots \\ \lambda m_1 & \lambda m_2 & \dots & m_0 \end{pmatrix} \begin{matrix} \leftarrow M \\ \leftarrow X.M \bmod E \\ \\ \leftarrow X^{n-1}.M \bmod E \end{matrix}$$

From Minkowski we can expect $\|\mathcal{M}\|_1 \approx (n!p)^{\frac{1}{n}}$.
We need $\phi > \|\mathcal{M}\|_1$ for parameter consistency so n will grow too much for large p.
In 2023, Meloni et al. suggest we take $\phi = 2^{128}$ instead of $\phi = 2^{64}$.

Size Comparison



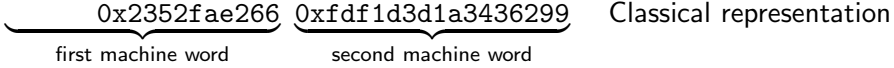
Results on large sizes



How to fix this?

Reduced coefficients in computer arithmetic

If we need to store `0x2352fae266fdf1d3d1a3436299` in memory:



Classical representation

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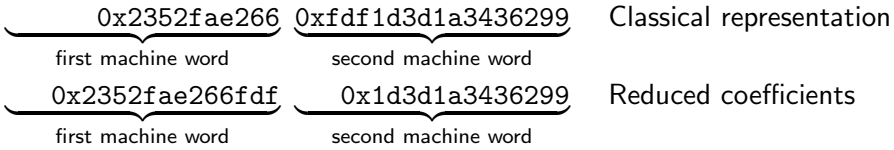


This lets us postpone carry propagation to the very end.

Montgomery CIOS

Algorithm 3 Montgomery CIOS Modular Multiplication

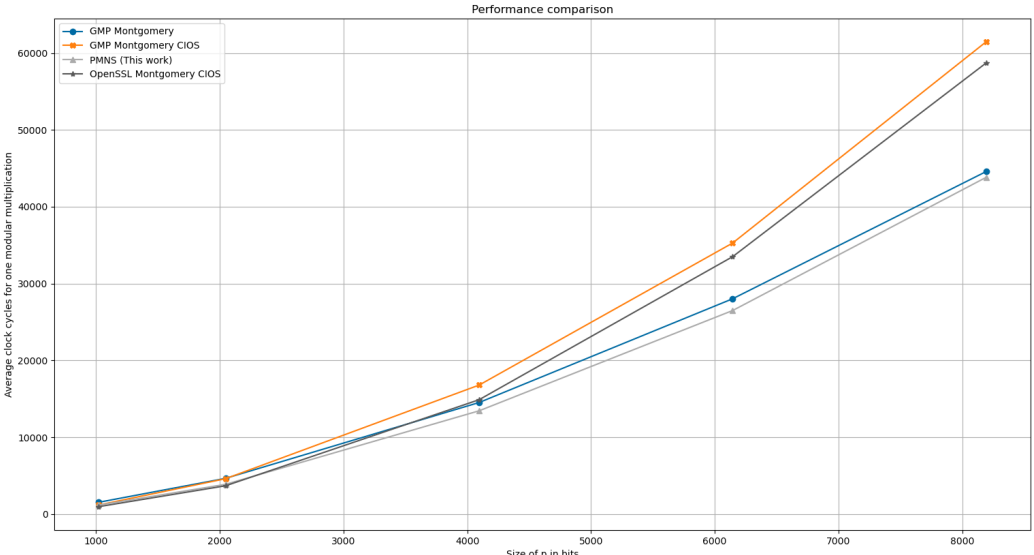
Require: m the modulus, $a \in \mathbb{Z}/m\mathbb{Z}$, $b \in \mathbb{Z}/m\mathbb{Z}$, $\phi = (2^{64})^h$ with $h \in \mathbb{N}^*$ such that $(2^{64})^{h-1} < m < (2^{64})^h$.

Ensure: $c \equiv ab\phi^{-1} \bmod m$, with $c \in \mathbb{Z}/m\mathbb{Z}$

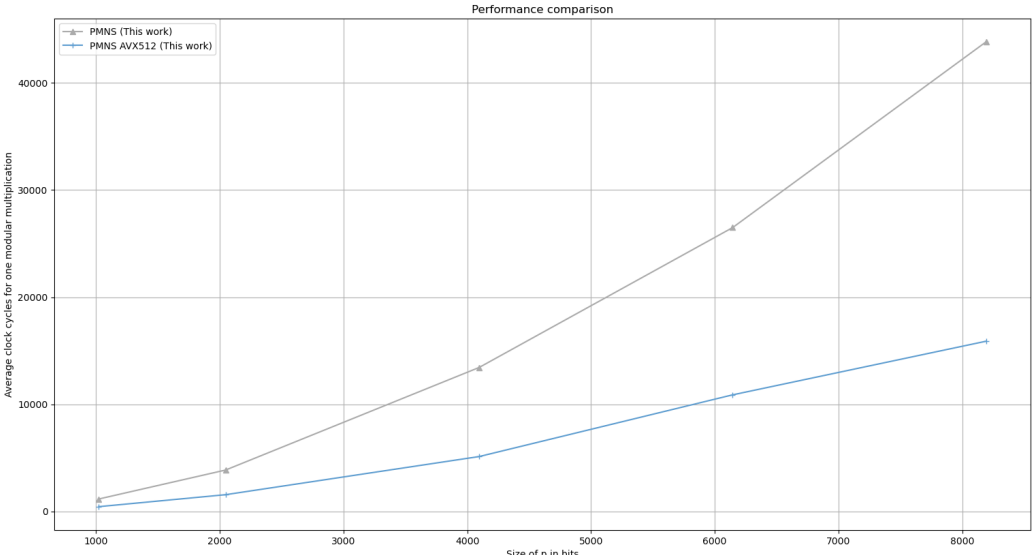
1: $c \leftarrow 0$
2: **for** $i = 0..h-1$ **do**
3: $\quad c \leftarrow c + a \times (\left\lfloor \frac{b}{(2^{64})^i} \right\rfloor \bmod 2^{64})$
4: $\quad q \leftarrow c \times m' \bmod 2^{64}$
5: $\quad c \leftarrow (c - q \times m)/\phi$ # Exact division
6: **end for**
7: **return** c

We only need the lower 64 bits of m^{-1}.

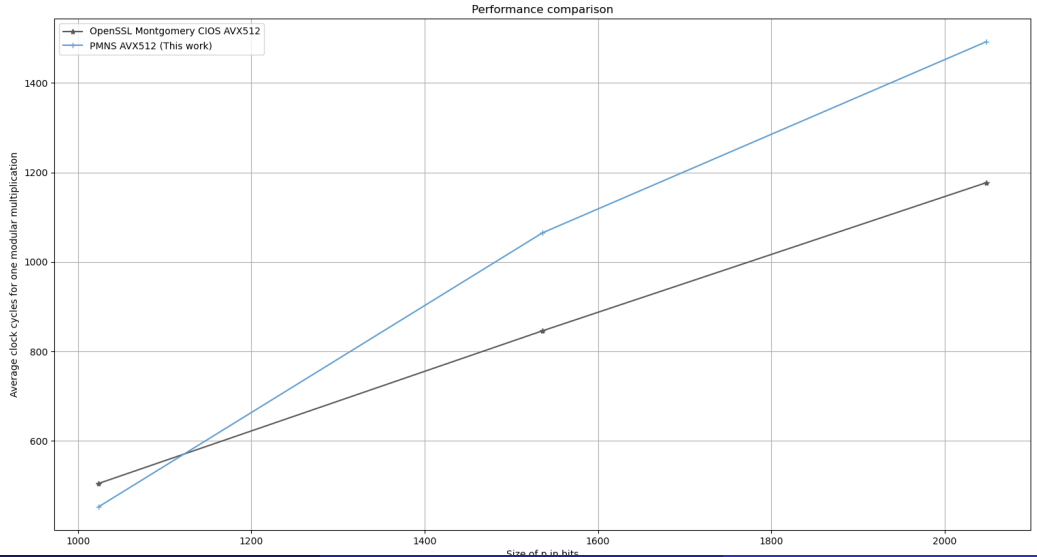
Results


Performance comparison
GMP Montgomery
GMP Montgomery CIOS
PMNS (This work)
OpenSSL Montgomery CIOS
Average clock cycles for one modular multiplication
Size of n in bits
0
10000
20000
30000
40000
50000
60000
1000
2000
3000
4000
5000
6000
7000
8000

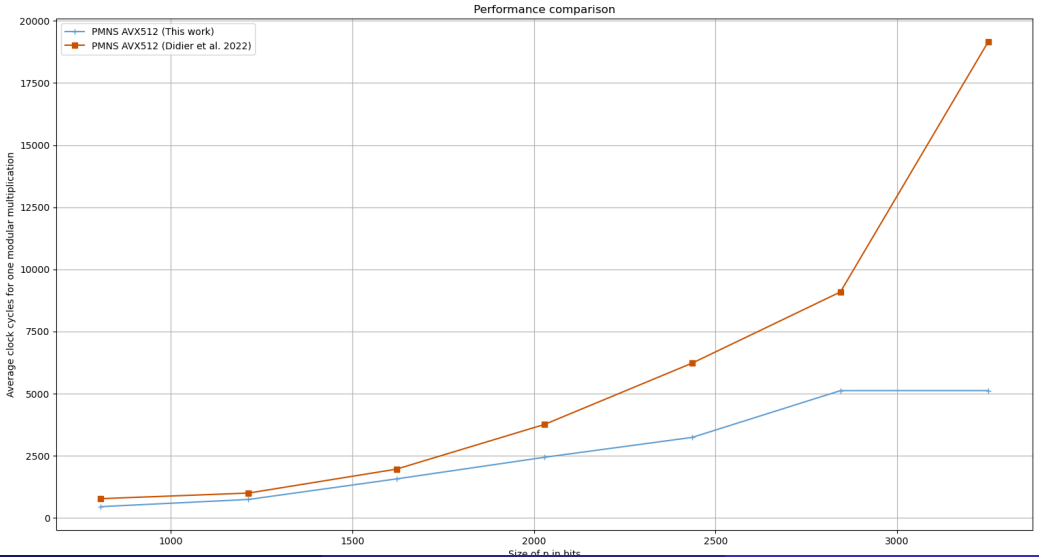
AVX512 vs Sequential



AVX512 vs OpenSSL



AVX512 vs Previous work



Thank you

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